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IR Renormalons and Fractional Instantons in SUSY Gauge Theories

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abstract

We study IR-renormalon divergences in $N = 1$ supersymmetric Yang Mills gauge theories and in two dimensional non linear sigma models with mass gap. We derive, in both types of theories, a direct connection between IR- renormalons and fractional instanton effects. From the point of view of large N dualities we work out a connection between IR-renormalons and $c = 1$ matrix models.

1 Introduction

Quantum Field Theory is defined in terms of perturbative expansions in the coupling constant. Even for asymptotically free theories and independently how small is the coupling constant these perturbative expansions are, for theories with no massive fields, divergent. To study the nature and meaning of these divergences is the most natural perturbative window into non perturbative physics.

In the large N limit [1] the divergences of the perturbative planar expansion are less severe due to the fact that the number of Feynman diagrams at n- loop order only grow geometrically [2]. This in particular means that instanton divergences are absent in the large N planar limit. However in the planar limit other type of divergences survive. These are known as renormalons [3], [4], [5] and can be associated with n-loop Feynman diagrams that behave thenselves as $n!$. Mathematically IR-renormalons manifest as singularities of the Borel transform of Green functions. These singularities prevent Borel summability in a direct way. Some time ago 't Hooft suggested that IR- renormalons could be important in the confinement dynamics and in the generation of a mass gap.

On the other hand the study of the dynamics of $N = 1$ super Yang Mills seems to point out to some sort of *"fractional instanton"* as the the dynamical origin of mass generation as well as of non vanishing chiral condensates. A fractional instanton is a formal concept and unfortunately we dont have, in the infinite volume limit, classical gauge configurations with fractional topological number [1]. However it is clear why fractional instantons could be relevant, namely and contrary to the case of instantons they survive in the large N planar limit.

In this paper we will provide considerable evidence that for $N = 1$ gauge theories *fractional instantons are in fact IR renormalons*. The way we will proceed is by evaluating the Borel transform in a stationary phase approximation. As a check we will work out some two dimensional non linear sigma models where the existence of a mass gap can be proved in the large N approximation.

An intimately related question is the already old discussion on the θ dependence of physics in the large N limit [7], [8]. The mass of the η' as well as the topological susceptibility for pure Yang Mills are examples of effects that in finite N we could associate with instantons but that we expect survive in the large N limit at order $\frac{1}{N}$. The understanding of these effects in the large N is very much based on our experience with two dimensional sigma models with dynamical generation of a mass gap [9], [10]. As already mentioned we will work out some IR- renormalon effects for these models and we will discuss their relevance in the mass gap generation.

A different approach to the physical meaning of renormalon divergences for the planar

[1]In the finite volume we have toron solution [6]

perturbative expansion is in the context of large N dualities [1]. For a gauge theory based on a gauge group of rank N the pertubative expansion is organized as a genus string expansion: $\sum_l g_s^{2l-2}F_l(t)$ where l is the genus, $t = g^2N$ is 't Hooft coupling and where the "string" coupling g_s is g^2. The dual string hypothesis is that in the large N the perturbative loop expansion in 't Hooft coupling t can be summed and interpreted as the closed string amplitude in some target space time characterized by t. For asymptotically free theories we should expect divergences of $F_0(t)$, governed by IR renormalons, independently how small is t. In some recent papers [11] and for simple traget space geometries related with the conifold, it was suggested that singularities at $t = 0$ could be related with fractional instanton effects. We observe that for asymptotically free theories the uncertainties due to renormalon (and instanton) divergences in the perturbative expansion can be interpreted as associated with a hidden gravitational sector in the sense of Matrix $c = 1$ models. This can be related with t' Hooft old suggestion that asymptotic freedom is not enough due to the uncertainties in the perturbative expansion even for arbitrarily small coupling. Probably the new ingredient we need in order to fix these uncertainties is intimately related with a hidden gravitational sector in asymptotically free gauge theories.

The plan of the paper is as follows. In the first section we mainly review some aspects of the use of instantons in supersymmetric QCD and we point out on the difficulties for extending this type of approach in the large N. In the second section we present our approach to IR renormalons and their connection with fractional instantons. In the next section we work out some two dimensional sigma models with mass gap. Finally we make some comments on the stringy interpretation in the context of large N dualities ending with a short remark on the connection of renormalons and Matrix models.

2 Instantons and $N = 1$ super Yang Mills

Asymptotically free theories with no infrared fixed point are expected to generate a mass gap dynamically. In the presence of massless fermions this is expected to be accompanied by chiral symmetry breaking. Unfortunately, the analytic tools available for studying these phenomena are rather limited. In the 80's and 90's considerable progress has been made, however, in supersymmetric gauge theories - thanks to holomorphy and nonrenormalization properties.

One important example is $N = 1$ chromodynamics [12], [13] with gauge group $SU(N_c)$ and with N_f flavors and $N_c \geq N_f + 1$. From general arguments it follows that nonperturbatively the theory generates a superpotential. This is a F term whose form is

$$F = bg^4\Lambda_{SQCD}^{\frac{3N_c-N_f}{N_c-N_f}} \int d^2\theta(\det_{rr'}\bar{Q}_{ir}Q^{ir'})^{-\frac{1}{N_c-N_f}} \tag{1}$$

where Λ_{SQCD} is the scale generated by the beta function. At one loop this is given in terms of a cutoff Λ_0 and the bare coupling g by

$$\Lambda_{SQCD} = \Lambda_0 e^{-\frac{8\pi^2}{(3N_c - N_f)g^2}} \tag{2}$$

At the classical level the theory has moduli given in terms of the expectation values of the squarks. We will restrict our attention to the case where the expectation values are symmetric in flavor indices and denoted by v. Then F-term implies a mass for the fermions

$$m \sim \Lambda_{SQCD}^{\frac{3N_c - N_f}{N_c - N_f}} \; v^{-\frac{2N_c}{N_c - N_f}} \sim \Lambda_0^{\frac{3N_c - N_f}{N_c - N_f}} \; v^{-\frac{2N_c}{N_c - N_f}} \; e^{-\frac{8\pi^2}{(N_c - N_f)g^2}} \tag{3}$$

Because of the F-term, the flat direction is however broken and in fact the true minimum is at infinity.

To make sense of the theory it is customary to add an explicit mass term with mass parameter M to the action. Then the total potential has a local minimum at

$$v = [\frac{g^4 N_f}{M(N_c - N_f)}]^{\frac{N_c - N_f}{2N_c}} \; \Lambda^{\frac{3N_c - N_f}{2N_c}} \tag{4}$$

This means that one has a well defined calculation for the gluino condensate $< \lambda\lambda >$ by using the Konishi anomaly

$$< \lambda\lambda >= \frac{1}{g^2} M \; v^2 \tag{5}$$

which leads to, using (4)

$$\begin{aligned} < \lambda\lambda > \; & \sim \; \frac{g^2 N_f}{N_c - N_f} \Lambda^{\frac{3N_c - N_f}{N_c - N_f}} \; v^{-\frac{2N_f}{N_c - N_f}} \\ & \sim \; (\frac{N_f}{N_c - N_f})^{\frac{N_c - N_f}{N_c}} \; M^{\frac{N_f}{N_c}} \; \Lambda^{\frac{3N_c - N_f}{N_c}} \end{aligned} \tag{6}$$

In fact these results may be used to learn about supersymmetric Yang-Mills theory *without* any matter. This is obtained as a limit of super-QCD by taking a limit $M >> \Lambda_{SQCD}$. One can then take M to be the cutoff scale at which the bare couplings ared defined. At energies much smaller than the cutoff the matter decouples leaving a pure super Yang-Mills theory. Using the one loop beta functions of the two theories one readily derives the following expression for the dynamically generated scale of the super-Yang-Mills, Λ_{SYM} is given by

$$\Lambda_{SYM} = (M^{N_f} \Lambda_{SQCD}^{3N_c - N_f})^{\frac{1}{3N_c}} \tag{7}$$

The expression (6) then becomes

$$< \lambda\lambda >= \Lambda_{SYM}^3 \tag{8}$$

which is now an expression in the pure Yang-Mills theory.

While the results above are quite general and valid for any N_f, N_c with $N_c \geq N_f - 1$, and follows from general symmetry properties together with holomorphy, not much is known about whether some specific configurations of gauge fields are primarily responsible for these expressions, except when $N_c = N_f - 1$. For $N_f = N_c - 1$ instantons are, in fact responsible for mass gap, chiral symmetry breaking and gluino condensation.

The instanton calculations are performed in the absence of the explicit mass term and expressed in terms of the running coupling constant $g^2(v)$ defined at the scale set by the modulus. This is related to Λ_{SQCD} by

$$\Lambda_{SQCD} = v\ e^{-\frac{8\pi^2}{(3N_c - N_f)g^2(v)}} \tag{9}$$

which is valid for any N_c, N_f with $N_c > N_f$. The expression for the fermion mass following from the superpotential may be the written as

$$m \sim v\ e^{-\frac{8\pi^2}{g^2(N_c - N_f)}} \tag{10}$$

A standard dilute gas computation using constrained instantons would however lead to

$$m \sim v\ e^{-\frac{8\pi^2}{g^2}} \tag{11}$$

which would agree with (10) if $N_c = N_f + 1$. Indeed, as is well known, for this case instanton methods are reliable, provided one always has $v >> \Lambda_{SQCD}$.

We will be interested in the large N_c limit where $g \to 0, N \to \infty$ with $g^2 N_c =$ fixed. In this limit, instantons have a large $O(N)$ action and do not contribute. It may appear puzzling, however, that the final results for $< \lambda\lambda >$ for the supersymmetric Yang-Mills theory (equation (7), (8) remain finite in this limit. However a closer examination quickly reveals that the relations above imply that $v \sim \Lambda_{SQCD}$ at large N, which invalidates the instanton calculation.

What, then, is the mechanism for dynamical mass generation and gluino condensation in the large-N limit ? A similar question can be asked in other asymptotically free theories where instanton physics is known to account for dynamical behavior at finite N.

Many years ago, 't Hooft showed that asymptotically free field theories have *infrared renormalon* singularities in the Borel transform of Green's function and suggested that these singularities play an important role in mass generation. Typically the Borel transform for these theories has a rich singularity structure. Instantons, for example, manifest themselves as singularities on the positive real axis. IR renormalons - whose location are determined by the beta function - are distinct from these. Indeed, in the large-N limit the instanton singularities are absent while the IR renormalons survive.

3 Fractional Instantons and IR renormalons

Let us start with a brief review of t Hoof's standard approach to instantons [14]. For simplicity we will first consider the case of pure Yang Mills without fermions. The partition function is

$$Z(g^2) = \int dA e^{-\frac{1}{g^2}\int dx^4 L(A)} \tag{12}$$

where we assume that gauge fixing and ghost terms has been included. Up to numerical factors what we get after taking into account the zero modes is (for $SU(2)$ gauge theory)

$$Z(g^2) = \int dx^4 \frac{d\rho}{\rho^5}\frac{1}{g^8} e^{-\frac{8\pi^2}{g^2(\frac{1}{\rho})}} \tag{13}$$

where now the integration over x correspond to the moduli of translations and the integration over ρ to the moduli of dilatations. The factor $\frac{1}{g^8}$ comes from the 8 zero modes. From (13) we get the effective action

$$L_{eff}(x; g^2) dx^4 = dx^4 \int \frac{d\rho}{\rho^5}\frac{1}{g^8} e^{-\frac{8\pi^2}{g^2(\frac{1}{\rho})}} \tag{14}$$

In general for $SU(N)$ we get in (14) a factor

$$\frac{1}{g^{4Nk}} \tag{15}$$

for $k = 1$ the instanton number. Notice that the effective lagrangian (14) is a dimension 4 operator as it should be.

Let us now formally consider "fractional instantons" of topological number $k = \frac{1}{N}$. If we extend to fractional topological charge the index theorem we get as the number of zero modes $4\frac{1}{N}N$ i.e just the four translations, independently of the rank of the gauge group. Notice that for topological charge $\frac{1}{N}$ we have not dilatation zero modes. Extending to this case the instanton result we get

$$dx^4 \frac{1}{g^4} e^{-\frac{8\pi^2}{g^2(\mu)N}} \mu^{\frac{11}{3}} \tag{16}$$

Contrary to the instanton case we don't get naturally a dimension four operator and therefore we should include some extra scale factor in order to get an operator that can qualifies as an effective action [2]. Notice also that the problem appears because the fractional instanton has not dilatation zero modes. All these problems in addition to the crucial fact that we have not any concrete classical solution with fractional topological number makes quite problematic the interpretation of fractional instanton effects in a semiclassical approximation.

[2] It is amussing to notice that the problem with dimensions is in a certain sense very small and 12 instaed of 11 will do the job.

The situation becomes a bit better if we have $N = 1$ supersymmetry. In this case we should take into account the fermionic zero modes for the gluino. We can expect an effective action

$$L_{eff}(g^2)dx^4d\theta^2 = dx^4d\theta^2 N\frac{1}{g^4}\Lambda^3 e^{-\frac{8\pi^2}{g^2(\frac{1}{\Lambda})N}} \tag{17}$$

that now as it is an operator of dimension 3 qualifies as an F term effective supersymmetric lagrangian. This is in fact the reason fractional instantons are normaly invoqued in the dynamics of $N = 1$ super Yang Mills. It is interesting to notice, already from the previous discussion, that fractional instantons are naturally related with supersymmetry. This is not very surprising since we know that due to supersymmetric Ward identities the gluino condensate associated with instanton effects can be -if formal cluster arguments are invoqued- factorized into fractional instanton contributions.

3.1 Renormalons

3.1.1 Borel Transform and Classical Configurations

As pointed out by t Hooft we can formally relate the partition function (12) with a Borel transform. In fact let us formally define a new variable z by $S(A) = z$ and let us denote $A(z)$ the corresponding gauge configuration. We get

$$Z(g^2) = \int_M \int_0^\infty dz(\frac{\partial S}{\partial z})_z^{-1} e^{-\frac{z}{g^2}} \tag{18}$$

where $\int_M$ represents the integral over the moduli of unequivalent gauge configurations $A(z)$ with $S(A) = z$. Comparing with a Borel transform

$$Z(g^2) = \int_0^\infty dz F(z) e^{-\frac{z}{g^2}} \tag{19}$$

it is clear that the Borel transform $F(z)$ is singular at the classical instanton solution for $z = 8\pi^2$. In general we will get singularities in the Borel plane for any classical configuration. Notice that we are working with euclidean signature and that the only relevant singularities for the Borel transform are classical euclidean configurations with $S(A) = z$ positive.

3.1.2 IR-Renormalons

As it is well known renormalons are divergences of perturbative expansions where the n loop contribution grows as $n!$. This $n!$ comes from the contribution of a n loop diagram itself and not as it is the case for instanton divergences from the number of diagrams contributing at the n loop order. This is the reason renormalons survive in the large N limit where it is known that the number of diagrams at order n grows at most geometrically. Let us start considering

the following formal perturbative expansion,

$$G(g^2) = \sum_{n=0}^{n=\infty} a_n g^{2n} \tag{20}$$

where

$$a_n = a^n n! \tag{21}$$

for some coefficient a that we will discuss in a moment. The Borel transform of (20) is

$$G(g^2) = \frac{1}{g^2} \int_0^\infty dz \sum_{n=0}^{n=\infty} \frac{a_n}{n!} z^n e^{-\frac{z}{g^2}} \tag{22}$$

Using (22) we observe that the renormalon divergence is at $z = \frac{1}{a}$. Notice that we are considering the existence of a g-independent piece in $G(g^2)$ given by a_0.

Now we will proceed to approximate (22) using stationary phase approximation. In order to do that we rewrite (22) as

$$G(g^2) = \int_0^\infty dz e^{f(z) - \frac{z}{g^2}} \tag{23}$$

The stationary phase approximation is given by

$$G(g^2) = e^{f(z(g^2)) - \frac{z(g^2)}{g^2}} \tag{24}$$

where $z(g^2)$ is the solution to $f'(z) = \frac{1}{g^2}$. Using that

$$\sum_{n=0}^{n=\infty} a^n z^n = e^{f(z)} \tag{25}$$

we get

$$z(g^2) = \frac{1}{a} - g^2 \tag{26}$$

and therefore

$$G(g^2) \sim \frac{1}{g^4} \frac{1}{a} e^{-\frac{1}{g^2 a}} \tag{27}$$

Notice from (26) that the renormalon singularity $z = \frac{1}{a}$ correspond to $g^2 = 0$. This is already telling us that renormalon divergences are important at weak coupling. Let us now discuss the physical meaning of renormalons. As we have already mention the renormalon singularity comes from the contribution of diagrams that behave as $n!$. In order to define the renormalon we will start considering the insertion of chains of vacuum bubbles into a propagator $P(k^2)$. Denoting the result $P_R(k^2)$ we get

$$P_R(k^2, \Lambda) = \sum_n \frac{1}{k^2} (ln(\frac{\Lambda^2}{k^2}))^n C^n \tag{28}$$

where $C = -\frac{\beta_1}{2} g^2$ and β_1 is the one loop beta function. We will be interested in IR renormalons i.e low momentum $k < \Lambda$ for Λ a cutoff.

The IR renormalon will be defined as

$$G_\alpha(g^2) = \int_0^\Lambda d^4k P_R(k^2) k^{2(1-\alpha)} = \sum_n \int_0^\Lambda d^4k \frac{1}{k^{2\alpha}} (ln(\frac{\Lambda^2}{k^2}))^n C^n \tag{29}$$

Changing variables $ln(\frac{1}{k'^2}) = x$ with $k' = k\Lambda$ we get

$$G_\alpha(g^2) = \sum_n \Lambda^{4-2\alpha} \frac{1}{(2-\alpha)} \frac{C^n}{(\alpha-2)^n} n! \tag{30}$$

with $\alpha < 2$. Denoting

$$a = \frac{-\beta_1}{2(\alpha-2)} = \frac{\beta_1}{d} \tag{31}$$

with $d = 4 - 2\alpha$ we get

$$G_\alpha(g^2) = \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} a^n n! g^{2n} \tag{32}$$

In terms of the Borel transform we get

$$G_\alpha(g^2) = \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} \frac{1}{g^2} \int_0^\infty dz F(z) e^{-\frac{z}{g^2}} \tag{33}$$

for $F(z) = \sum_n a^n z^n$.

Notice that

$$G_\alpha(g^2) = \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} + \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} a g^2 + ... \tag{34}$$

If we substract the g- independent piece $\frac{\Lambda^{4-2\alpha}}{(2-\alpha)}$ we obtain

$$G^S_\alpha(g^2) = \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} \int_0^\infty dz F(z) e^{-\frac{z}{g^2}} \tag{35}$$

for $G^S_\alpha(g^2)$ the substracted Green function.

Let us now consider the meaning of $G_\alpha(g^2)$. We can associate $P_R(k^2)$ with

$$< \Phi(k)\Phi(-k) > \tag{36}$$

where Φ represents a local quantum field of dimension one and where we are summing all vacuum bubble chains insertions. In this sense $G_{\alpha=1}(g^2) = \int_0^\Lambda d^4k < \Phi(k)\Phi(-k) >$ and generically for $\alpha = 1 - n$ with $\alpha < 2$

$$G_\alpha(g^2) = \int_0^\Lambda d^4k < \partial_1..\partial_n \Phi(k) \partial_1...\partial_n \Phi(-k) > \tag{37}$$

where it is implicit the condition of Lorentz scalar and gauge singlet.

Using (31) and (33) we get for $\alpha = 1 - n$ singularities in the Borel plane at

$$z = \frac{2+2n}{\beta_1} \tag{38}$$

We can now use the stationary phase approximation in order to estimate $G_\alpha(g^2)$, the result is

$$G_\alpha(g^2) = \frac{\Lambda^{4-2\alpha}}{(2-\alpha)} \frac{1}{g^4} \frac{1}{a} e^{-\frac{1}{g^2 a}} \tag{39}$$

Until now we have been working in four dimensions. The generalization to two dimensions is quite simple. We get

$$G_\alpha(g^2) = \frac{\Lambda^{2-2\alpha}}{(1-\alpha)} \frac{1}{g^4} \frac{1}{a} e^{-\frac{1}{g^2 a}} \tag{40}$$

where now $\alpha < 1$. In next section we will use these expression in the analysis of two dimensional non linear sigma models.

Finally we will extend our computations to the case in which we have $N = 1$ supersymmetry [3]. The result in four dimensions is

$$G_{\hat{\alpha}}(g^2) = \frac{\Lambda^{4-2\hat{\alpha}}}{(2-\hat{\alpha})} \frac{1}{g^4} \frac{1}{a} e^{-\frac{1}{g^2 a}} \tag{41}$$

where as before $a = \frac{\beta_1}{d}$, $d = 4 - 2\hat{\alpha}$ and

$$\hat{\alpha} = \alpha + \frac{1}{2} \tag{42}$$

with $\alpha < 2$. For $\alpha = 0$ this corresponds to $d = 3$. For $N = 1$ super Yang Mills with

$$\beta_1 = \frac{3N}{8\pi^2} \tag{43}$$

i.e $a = N$ we get

$$G_{\hat{\alpha}=\frac{1}{2}}(g^2) = \Lambda^3 \frac{1}{g^4} \frac{8\pi^2}{N} e^{-\frac{8\pi^2}{g^2 N}} \tag{44}$$

that we can interpret as $< \lambda\lambda >$ gaugino condensate.

In the SUSY case the singularities in the Borel plane for $\hat{\alpha} = \frac{3}{2} - n$ are at

$$z = \frac{1+2n}{\beta_1} \tag{45}$$

3.1.3 Fractional Instanton versus IR Renormalon

Before reaching the main conclusion of this section concerning the connection of IR renormalons and fractional instantons we must come back to our computation of $G_\alpha(g^2)$ and to fix the factors of N. In the case of $N = 1$ super Yang Mills with all the fields in the adjoint representation it is natural to associate with $G_\alpha(g^2)$ a factor N^2. Thus we must modify (44) to

$$G_{\hat{\alpha}=\frac{1}{2}}(g^2) = \Lambda^3 N \frac{1}{g^4} \frac{16\pi^2}{3} e^{-\frac{8\pi^2}{g^2 N}} \tag{46}$$

[3] In order to do that we can start replacing $P_R(k^2)$ in (29) by a superpropagator.

Now we can compare this result with the fractional instanton result (17) we formally got by extending the index theorem to the case of fractional topological number. We observe that up to numerical factors we have not fixed in the formal instanton computation both nicely coincide. In this way we can conclude the following result:

For $N = 1$ super Yang Mills the contribution of the first IR renormalon corresponding to the Borel singularity at $z = \frac{3}{3N} = \frac{1}{N}$ is, once we estimate the Borel transform in the stationary phase approximation, equivalent to a fractional instanton of topological number $k = z = \frac{1}{N}$.

Notice that in the $N = 1$ case the IR renormalon contribution produces the right powers of g, namely $\frac{1}{g^4}$ consistent with the dimension of the moduli for topological charge $\frac{1}{N}$ [4].

3.1.4 Comments on Borel Summability

In all the previous examples the IR renormalon singularity at $z = \frac{1}{a}$ prevents ,a priori, to integrate in the Borel-Laplace transform from $z = 0$ to $z = \infty$. In fact the region where $F(z)$ is convergent is $z < \frac{1}{a}$ which is certainly not enough to prove Borel summability. In our discussion above we have decided to estimate the Borel-Laplace transform using a phase stationary approximation. This means that we have saturated the integral over z by the saddle point contribution. Recall from (26) that the saddle point is given by

$$z = \frac{1}{a} - g^2 \tag{47}$$

interpreted as a function $z(g^2)$. Thus the contribution of the IR renormalon $z = \frac{1}{a}$ can be obtained by taking the limit $g \to 0$ in the phase stationary result. It is instructive to see that in this limit the leading contribution comes from the exponential factor $e^{-\frac{8\pi^2}{g^2 N}}$ (in the case of $N = 1$ SUSY Yang Mills) that we can interpret as the fractional instanton. Moreover this exponential fractional instanton factor smooths the divergence. At this point it is natural to ask why not to do the same for pure Yang Mills. Of course in pure Yang Mills we have IR renormalons as well. The first one that can be associated with a gauge invariant operator is at $z = \frac{4}{\beta_1}$ corresponding to $d = 4$ and with $\beta_1 = \frac{11N}{3}$. The exponential factor in this case will becomes

$$e^{-\frac{8\pi^2 12}{g^2 11N}} \tag{48}$$

which unfortunately can not be interpreted as a fractional instanton [5]. All this seems to indicate that something dynamically very special take place when *the IR renormalon contribution can be interpreted , using a formal extension of index theorems, as a fractional instanton.* It is this coincidence what seems to be at the core of the magic of supersymmetry.

[4] Recall that the dimension of the moduli is $4Nk$ for k the topological number

[5] Formally it could be interpreted as a fractional instanton of topological number $\frac{12}{11N}$. Using index theorem this will produce a dimension of moduli $4N\frac{12}{11N} = \frac{46}{11}$ which is certainly non sense.

3.1.5 Remark: On the η' mass

In reference [7] it was assumed that for pure Yang Mills without fermions physics depends on the θ vacuum parameter to order $\frac{1}{N}$ and not to order e^{-N} as it is suggested by pure instanton effects. A direct consequencce of this assumption, that was based on the solution in planar limit of two dimensional non linear sigma models [9] [10], is a mass formula ,in the large N limit, for the ninth light pseudoscalar, the η' [7][8]. In fact in the presence of massless fermions physics should be independent of θ therefore in order to cancell the θ dependence of pure Yang Mills we need a pseudoscaler of mass $O(\frac{1}{N})$.

The crucial quantity is the topological susceptibility that directly measures the θ dependence of the vacuum energy of pure Yang Mills

$$U(k) = \int d^4 x e^{ikx} < T(FF^*(x)FF^*(0) > \tag{49}$$

in the limit $k = 0$. This quantity is intimately related with IR renormalons. In fact we have

$$< T(FF^*(x)FF^*(x) >= \int_0^\Lambda d^4 k U(k) = \sum_n \int d^4 k k^4 (ln(\frac{\Lambda^2}{k^2}))^n C^n \tag{50}$$

with C fixed by the beta function as usual. This is essentially equivalent to what we have denoted $G_\alpha(g^2)$ for $\alpha = -2$. The two dimensional version is

$$< T(FF^*(x)FF^*(x) >= \int_0^\Lambda d^2 k U(k) = \sum_n \int d^2 k k^2 (ln(\frac{\Lambda^2}{k^2}))^n C^n \tag{51}$$

In both cases we can estimate this quantity using stationary phase approximation. The result in four dimensions for Yang Mills is

$$< T(FF^*(x)FF^*(x) >\sim \Lambda^8 \frac{1}{g^4} \frac{8\pi^2 24}{11N} e^{-\frac{8\pi^2 24}{11Ng^2}} \tag{52}$$

This of course is not exactly what we need in order to get the mass difference for the η', that is given in terms of $U(k=0)$. On the other hand using low energy theorems the following mass formula was derived in [15]

$$m^2_{\eta'} f_{\eta'} \sim < F(x)F(x) > \tag{53}$$

and we can try to estimate $< F(x)F(x) >$ as $G_{\alpha=0}(g^2)$, which is a good indication on the potential contributions of IR renormalons to the η' mass.

4 VY Effective Lagrangians and Two Dimensional Non Linear Sigma Models

4.1 Non Supersymmetric Case: The $O(N)$ Model

The large-N limit of the two dimensional $O(N)$ sigma model provides a simple example of the IR renormalon. The model has N vector fields σ^a with a lagrangian

$$L = \frac{1}{2g^2}\sum_{a=1}^{N}(\partial\sigma^a)(\partial\sigma^a) \tag{54}$$

with the constraint

$$\sum_a \sigma^a\sigma^a = 1 \tag{55}$$

This model can be exactly solved in the large-N limit by introducing a lagrange multiplier field $\lambda(x)$ and rewriting the lagrangian as

$$L = \frac{1}{2g^2}\sum_{a=1}^{N}(\partial\sigma^a)(\partial\sigma^a) + \frac{N\lambda}{2}(\sum \sigma^a\sigma^a - 1) \tag{56}$$

Integrating out the σ fields now leads to an effective lagrangian for the field λ

$$L = \frac{N}{2}\mathrm{Trlog}(-\frac{\partial^2}{g^2} + N\lambda) - \frac{N\lambda}{2} \tag{57}$$

At large-N, the functional integral over λ is saturated by a translationally invariant saddle point λ_0 which satisfies the gap equation

$$\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2 + g^2N\lambda_0} = \frac{1}{g^2N} \tag{58}$$

whose solution is

$$m^2 \equiv g^2N\Lambda_0 = \Lambda^2\ \exp[-\frac{4\pi}{g^2N}] \tag{59}$$

where Λ denotes an ultraviolet cutoff. Expanding around this saddle point shows that m^2 is in fact the dynamically generated mass of the field σ^a. This also leads to the exact beta function at $N = \infty$

$$\beta(g^2N) = \Lambda\frac{d}{d\Lambda}g^2N = -2\frac{1}{4\pi}(g^2N)^2 \tag{60}$$

The running coupling constant is then given by

$$g^2N(p^2) = \frac{g^2N}{1 + \frac{g^2N}{4\pi}\log(\frac{p^2}{\Lambda^2})} \tag{61}$$

Consider now the quantity

$$G^{ab}_{\mu\nu}(g^2N) = \int d(p^2) < \partial_\mu\sigma^a(p)\partial_\nu\sigma(-p) > \tag{62}$$

This is given by

$$G^{ab}_{\mu\nu}(g^2N) = -g^2\delta^{ab}\int d(p^2)\frac{p_\mu p_\nu}{p^2+m^2} \tag{63}$$

Using the expression for the mass gap this evaluates to

$$G^{ab}_{\mu\nu}(g^2N) = -\frac{g^2}{2}\delta^{ab}\delta_{\mu\nu}\Lambda^2[1-\frac{4\pi}{g^2N}e^{-\frac{4\pi}{g^2N}}] \tag{64}$$

Note that the first term is quadratically divergent while the second term, which comes because of the presence of a mass gap is logarithmically divergent. The latter can be seen by noting that the second term is

$$m^2\log(\frac{\Lambda^2}{m^2}) \tag{65}$$

Subtracting the quadratic divergence one gets

$$(G^R)^{ab}_{\mu\nu} = \frac{g^2}{2}\delta^{ab}\delta_{\mu\nu}\Lambda^2\ \frac{4\pi}{g^2N}e^{-\frac{4\pi}{g^2N}} \tag{66}$$

The perturbation expansion of the model is performed by solving the constraint explicitly

$$\sigma^N = \sqrt{1-\sum_i n^i n^i} \tag{67}$$

where we have renamed $\sigma^i = n^i$ for $i = 1\cdots N-1$. The action then has an infinite number of interaction terms

$$L = \frac{1}{2g^2}[\partial n^i\partial n^i + \frac{n^i n^j\partial n^i\partial n^j}{(1-n^i n^i)}] \tag{68}$$

and peforming the large-N expansion involves summation over an infinite set of bubble diagrams. It is clear, however, that the final result for the propagator is

$$< n^i(p)n^j(-p) >= \delta^{ij}\frac{g^2(p)}{p^2} \tag{69}$$

where $g^2(p)$ is the running coupling constant defined in (61).

Let us now compute the quantity $G^{ij}_{\mu\nu}$ as defined in (62) by using (69). The result is easily seen to be

$$G^{ij}_{\mu\nu} = \frac{g^2}{2}\delta^{ij}\delta_{\mu\nu}\Lambda^2\ \frac{4\pi}{g^2N}\int dy\ \frac{e^{-y}}{\frac{4\pi}{g^2N}-y} \tag{70}$$

It is now clear that the Borel transform $F_G(z)$ of this Green's function has a pole at

$$z = \frac{4\pi}{N} \tag{71}$$

This pole is the infrared renormalon. The integral in (70) is of course ill defined. If we *define* this integral as being dominated by the pole we get a result which is in agreement with (66). Since the latter is entirely due to mass generation, it is clear that the dynamical mass may be thought of being produced by the IR renormalon.

We can now easily compare with the discussion on IR renormalons in previous section. In fact we have

$$G^{ij}_{\mu\nu} = \delta^{ij}\delta_{\mu\nu}\int_0^{\Lambda} dk^2 \frac{g^2}{1+\frac{g^2 N}{4\pi}log(\frac{k^2}{\Lambda^2})} \tag{72}$$

which can be written as

$$\Lambda^2 \int_0^{\infty} dz F(z) e^{-\frac{z}{g^2}} \tag{73}$$

with

$$F(z) = \sum a^n z^n = \frac{1}{1-az} = \frac{1}{1-\frac{Nz}{4\pi}} \tag{74}$$

where $a = \frac{\beta_1}{d} = \frac{N}{2\pi.2}$. Since we are interested in the Green function after substracting the term of order $g^2\Lambda^2$ we should consider

$$\Lambda^2 g^2 \int_0^{\infty} dz F(z) e^{-\frac{z}{g^2}} \tag{75}$$

Evaluating this integral using the stationary phase approximation we get a result in *perfect agreement* with (66). This concludes the proof on the connection of IR renormalons and mass generation in the $O(N)$ non linear sigma model.

4.1.1 VY Effective Lagrangian

Let us denote λ the Lagrange multiplier. After gaussian integration we get

$$L_{eff}(\lambda) dx^2 = (-\frac{1}{g^2}\lambda + N logdet|| - \partial^2 + \lambda||) dx^2 \tag{76}$$

Denoting

$$F(\lambda) = logdet|| - \partial^2 + \lambda|| \tag{77}$$

we get

$$\frac{\partial F}{\partial \lambda} = \frac{1}{4\pi} log(\frac{\Lambda^2}{\lambda}) \tag{78}$$

which means

$$F(\lambda) = \frac{1}{4\pi}\lambda(log(\frac{\Lambda^2}{\lambda}) + 1) \tag{79}$$

thus

$$L_{eff}(\lambda) = -\frac{1}{g^2}\lambda + \frac{N}{4\pi}\lambda(log(\frac{\Lambda^2}{\lambda}) + 1) \tag{80}$$

Defining

$$\Lambda^2 = m^2 e^{\frac{4\pi}{Ng^2}} \tag{81}$$

for m the mass scale and Λ the cutoff, we get

$$L_{eff}(\lambda) = \frac{N}{4\pi}\lambda(log(\frac{m^2}{\lambda}) + 1) \tag{82}$$

that is the VY effective lagrangian [16] for this model.

It is interesting to notice that we have obtained a VY effective lagrangian in a model where instanton effects are manifestly absent.

4.1.2 VY Effective Lagrangian, Renormalization Group and Legendre Transform

For simplicity in notation let us define

$$x = \frac{1}{g^2} \tag{83}$$

In terms of this variable let us define the function

$$f(x) = \frac{N}{4\pi} m^2 e^{\frac{x}{N}} \tag{84}$$

Let us now denote λ the conjugated variable to x and let us perform the Legendre transform of $f(x)$, namely

$$Lf(\lambda) = \frac{N}{4\pi} \lambda (log(\frac{m^2}{\lambda}) + 1) \tag{85}$$

which is in fact the VY effective lagrangian (82). Recall that the Legendre transform $Lf(\lambda)$ of a function $f(x)$ is defined as

$$Lf(\lambda) = \lambda x(\lambda) - f(x(\lambda)) \tag{86}$$

for $x(\lambda)$ the solution to the Legendre equation

$$f'(x) = \lambda \tag{87}$$

What we observe is the following nice fact. The function f such that Lf is the VY effective action is just *the one such that the solution $x(\lambda)$ to equation (87) is the renormalization group running coupling with λ playing the role of the RG parameter.* In fact from (84) we get

$$x(\lambda) = \frac{1}{g^2}(\lambda) = \frac{N}{4\pi} log(\frac{\lambda}{m^2}) \tag{88}$$

recall that λ , the Lagrange multiplier is of dimension 2.

4.2 The supersymmetric Case: $N = 2$ Models

The supersymmetric CP^{N-1} model has been a typical toy model of four dimensional asymptotically free field theories as QCD and $N = 1$ super Yang Mills. Let us first briefly recall some well known facts about this model. The CP^{N-1} model is asymptotically free. The n particles and the fermionic superpartners ψ associted with the elementary fields get dynamically a mass. Chiral condensates $< \psi\bar{\psi} >$ get a vev with N different vacua in agreement with the value of $Tr(-1)^F$. There are solitons interpolating between the different vacua that are the massive n and ψ particles. The quantum chiral ring has the structure $x^N = 1$. The model possesses a $N = 2$ supersymmetry and the solitonic spectrum admits a $N = 2$ Landau Ginzburg description. The chiral anomaly is $\partial j^5_\mu = 2Ng^2 \epsilon_{\mu,\nu} \partial_\mu \bar{n} \partial_{nu} n$ which means that as well as it is the case for $N = 1$ super Yang Mills we can expect a η' of mass of order one.

In order to fix notation let us briefly review some generalities on two dimensional $N = 2$ models. We will denote Φ_i the chiral superfields and V the vector superfield. Under $U(1)$ gauge transformations

$$V \rightarrow V + i(\Lambda - \bar{\Lambda}) \tag{89}$$

with Λ a chiral superfield, and

$$\Phi_i \rightarrow e^{-iQ_i\Lambda}\Phi_i \tag{90}$$

for Q_i the $U(1)$ charges. We will reduce ourselves to abelian gauge group. It is convenient to introduce the superfield Σ as

$$\Sigma = \frac{1}{\sqrt{2}}\bar{D}_+ D_- V \tag{91}$$

We will denote σ the scalar component of this superfield. The lagrangian is given by

$$L = \sum_i \Phi_i e^{Q_i V}\Phi_i + W(\Phi) + \frac{1}{4e^2}\bar{\Sigma}\Sigma - rV \tag{92}$$

where W is the standard superpotential, e is the gauge coupling constant and $-rV$ is the Fayet Iliopoulos term with r a free parameter. We can also add to this lagrangian the usual θ term. The parameters r and θ combine into a complex variable t defined as

$$t = ir + \frac{\theta}{2\pi} \tag{93}$$

Using t the FI term and the θ term can be written as

$$i\frac{1}{2\sqrt{2}}t\Sigma + c.c \tag{94}$$

From now on we will consider for simplicity models with $W = 0$. Using the equations of motion we get

$$D = -e^2(\sum_i Q_i|\phi_i|^2 - r) \tag{95}$$

where D as usual is the last component of the vector superfield V andϕ_i the scalar components of the chiral superfields Φ_i. The classical potential is

$$U = \frac{1}{2e^2}D^2 + 2|\sigma|^2 \sum_i Q_i^2|\phi_i|^2 \tag{96}$$

Let us now consider the classical supersymmetric vacua defined by $U = 0$. If the FI coupling r is non vanishing the solution is $\sigma = 0$ and

$$\sum_i Q_i|\phi_i|^2 = r \tag{97}$$

Around this vacua the σ field becomes massive and the low energy lagrangian is simply the sigma model with target space defined by (97). If we chose $Q_i = 1$ and $\sum_i Q_i = N$ we get from

(97) the standard supersymmetric CP^{N-1} model. On the other hand if we consider $r = 0$ we get a diffeterent type of solution to $U = 0$, namely $\phi_i = 0$ and σ different from zero. Around this classical vacua the σ field is massless but the chiral superfields Φ_i become massive with a mass of the order of $|\sigma|$. The corresponding low energy physics would be obtained after integrating these massive fields [17], [18].

Let us briefly describe the integration of the massive fields. For the CP^{N-1} model with N chiral superfields and charges $Q_i = 1$ we get

$$\int d\phi_i exp - \int (D\phi_i D\bar{\phi}_i + D(|\phi_i|^2) + |\sigma|^2|\phi_i|^2) = exp - L_{eff}(D,\sigma) \tag{98}$$

Performing the integration we get

$$L_{eff}(D,\sigma) = \frac{1}{4\pi} N((D + |\sigma|^2) log(\frac{\Lambda^2}{(D + |\sigma|^2)}) + D + |\sigma|^2 \tag{99}$$

After adding the tree level term $-rD$, the equation of motion for the auxiliary field D is

$$\frac{1}{4\pi} N log(\frac{\Lambda^2}{D + |\sigma|^2}) - r = 0 \tag{100}$$

Now we can ask ourselves what is the effective action $L(\Sigma)$ that reproduces this equation of motion for the D field. It is easy to observe that for large $|\sigma|^2$ the F-term is given by

$$L_F(\Sigma) = d\theta^2 (\frac{N}{2\pi} N\Sigma(log(\frac{\Lambda}{\Sigma}) + 1) + \frac{it}{2\sqrt{2}}\Sigma) \tag{101}$$

with t at the scale Λ. This is the VY effective lagrangian as it should be expected.

For future use we will consider also the case of the $N = 2$ model associated to the conifold. In this case we have 4 fields with charges $+1, +1, -1, -1$. The effective action $L_{eff}(D,\sigma)$ in this case is

$$L_{eff}(D,\sigma) = \frac{1}{4\pi}(2(D + |\sigma|^2) log(\frac{\Lambda^2}{(D + |\sigma|^2)}) + 2(|\sigma|^2 - D) log(\frac{\Lambda^2}{|\sigma|^2 - D}) + 4|\sigma|^2) \tag{102}$$

After adding the tree level terms $-rD + D^2$ the equation of motion for D is

$$D + \frac{1}{4\pi}(2 log(\frac{(|\sigma|^2 - D)}{(D + |\sigma|^2)}) - r = 0 \tag{103}$$

Expanding in $|\sigma|^2$ we get

$$D(1 - \frac{1}{\pi|\sigma|^2}) = r \tag{104}$$

The F-term of the corresponding effective lagrangian $L(\Sigma)$ is just $it\Sigma$. This reproduces the equation $D = r$. In order to reproduce the correction $\frac{1}{|\sigma|^2}$ in (104) we need to add to the effective lagrangian a D-term of the type [19], [18]

$$log\Sigma log\bar{\Sigma} d\theta^2 d\bar{\theta}^2 \tag{105}$$

The same is true for the equation of motion (100) for CP^{N-1} model. Corrections of $O(\frac{D}{|\sigma|^2})$ to $log(\frac{\Lambda^2}{|\sigma|^2})$ generate an extra D-term as the one given in (105). These comments will become relevant in next subsection.

Notice from (102) that although the $N = 2$ model associated with the conifold is conformal invariant with vanishing beta function the effective lagrangian for σ if $D = 0$ is again a VY effective lagrangian.

As we did for the $O(N)$ model we can reproduce these results from the point of view of IR renormalons. In fact in this case we should just consider in two dimensions $G_{\hat{\alpha}=\frac{1}{2}}(g^2)$ with $\beta_1 = \frac{N}{4\pi}$. This IR renormalon contribution reproduces $< \Sigma >$. As before their Legendre transform reproduces the VY effective action.

4.2.1 Topological Susceptibility

As a final comment let us say few words on the topological susceptibility for the CP^{N-1} model. At finite N this model has instantons which leads to a θ dependence of the vacuum energy. In the $\theta = 0$ the signature of this fact is the behavior of the two point function of the topological charge density $T(x)$

$$T(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu \tag{106}$$

at zero momentum. It is clear that to any finite order in perturbation theory the quantity

$$\mathrm{Lim}_{p\to 0} < T(p)T(-p) > \tag{107}$$

vanishes since each T is a total derivative. However in the presence of instantons, a standard calculation based on dilute gas approximations leads to a θ dependence of the vacuum energy of the model, and hence to a nonvanishing topological susceptibility. The result goes as $\exp\left[-\frac{1}{g^2}\right]$

In the large-N limit, $N \to \infty$, $g \to 0$ with g^2N fixed instantons give a vanishing contribution and one might think that in this limit the vacuum energy becomes independent of θ. However, as has been shown in the 80's this conclusion is wrong. In a way entirely similar to the $O(N)$ nonlinear sigma model the lagrange multiplier acquires a nonzero vacuum expectation value. Furthermore, expanding around the saddle point one finds that the auxiliary gauge fields acquire a kinetic term

$$\frac{g^2N}{48\pi M^2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{108}$$

A straightforward calculation of the two point function of $T(p)$ then leads to the result

$$< T(p)T(-p) >= \frac{3M^2}{\pi g^2 N} \tag{109}$$

The crucial ingredient in this is the emergence of a kinetic term for the gauge field. The propagator then cancels the explicit power of momentum which comes from the definition of $T(p)$.

In the $N=2$ formalism, the dynamically generated kinetic term for the auxiliary gauge field is contained in the D-term (105) discussed above. Recall that we find this term when we consider effects of $O(\frac{D}{|\sigma|^2})$ which is the analog of expanding around the saddle point. From the IR renormalon point of view we were able to derive the F-term corresponding to VY effective lagrangian but not the D-term.

5 VY and large N Duality: Some Comments

5.1 The stringy meaning of VY effective Lagrangian

It is a well known result of the large N limit that the perturbative expansion of quantum field theory amplitudes for gauge theories is organized as a stringy genus expansion where the gauge coupling g^2 is playing the role of the string coupling. Generically we expect for the perturbative expansion the following structure

$$\sum_l (g^2)^{2l-2} F_g(t) \tag{110}$$

where $t = g^2 N$ is the t'Hooft coupling and l represent the genus. On the other hand we know that renormalon effects survive at large N and we expect that they are important contributions in the planar $l = 0$ limit. For $N = 1$ supersymmetric Yang Mills we have computed the renormalon contribution in section two (see equation (20)) and we have derived, using Borel transformation and stationary phase approximation, the following result for $d = 3$

$$G_{\hat{\alpha}=\frac{1}{2}}(g^2) = \Lambda^3 N^2 \frac{1}{g^4}\frac{8\pi^2}{N} e^{-\frac{8\pi^2}{g^2 N}} \tag{111}$$

If now we try to read this contribution as the genus zero contribution in the sense of (110) we observe that the counting of powers of g is just the right one and we will get

$$\frac{1}{g^2} F_0(t) \tag{112}$$

with

$$F_0(t) = \Lambda^3 N 8\pi^2 e^{-\frac{8\pi^2}{t}} \tag{113}$$

This is quite nice since it is exactly the Legendre transform of the VY effective lagrangian, where we take S as the conjugated variable.

In summary we observe two things. The first one is that the VY effective F term lagrangian is just the Legendre transform of the renormalon contribution to the genus zero amplitude. The second one is that the factor $\frac{1}{g^4}$ corresponding to the genus zero contribution is precisely the one fixed by the renormalon contribution and also the one you will expect, by naiv counting of zero modes (see discussion in section two) for a fractional instanton configuration of topological number $\frac{1}{N}$. Recall that in this case and independently of the rank of the gauge group we get just

the four translation zero modes. This seems to indicate that the so called fractional instantons have a very natural interpretation as a zero genus contribution.

There is a formal connection of VY with the $N = 2$ model associated with the conifold [11] as well as with the $c = 1$ string. If we consider the Laplace transform

$$F(S) = \int_0^\infty \frac{1}{\sigma^2} e^{-S\sigma} \tag{114}$$

this is divergent for $\sigma \sim 0$. This is exactly the same problem we find in the $c = 1$ case for the contribution of surfaces of small area [20] . The standard way to cure this divergence is by diferenciating with respect to S. After performing two derivatives we get

$$\frac{\partial^2 F}{\partial^2 S} = \frac{1}{S} \tag{115}$$

and therefore $F(S) = SlogS$ i.e the structure of VY effective lagrangian. The integrand in (114) can be on the other hand naturally associated with the $N = 2$ conifold model (see section 4.2) if we identify S with the FI coupling.

5.1.1 Toroidal Compactification

Using holomorphicity we can read the F-terms of $N = 1$ super Yang Mills from the F- terms of the $N = 2$ theory we obtain by compactification on T^2 [21]. For the case of $U(N)$ the two dimensional model is the $N = 2$ non linear sigma model with quantum chiral ring isomorphic to the chiral ring of the original four dimensional field theory, namely the $N = 2$ CP^{N-1} model we have briefly described in section 3. The Kahler class is identified with the Yang Mills coupling constant, and Σ with the “dimensional reduction” of the glueball field S.

The isomorphism of the chiral rings between the two models is quite clear at the level of the renormalon contributions. In four dimensions we get

$$\int \frac{d^4k}{k^{2\alpha_2}} (logk^2)^n C_4^n \tag{116}$$

with $C_4 = \frac{\beta_1}{2}$ for $\beta_1 = 3N$ In two dimensions

$$\int \frac{d^2k}{k^{2\alpha_2}} (logk^2)^n C_2^n \tag{117}$$

with $C_2 = N$. Clearly both are the same if in four dimensions we consider an operator of dimension 3 and in two dimensions an oparator of dimension 1. More precisely the condition for this dimensional reduction to work at the renormalon level is

$$\frac{d}{\beta_1^{D=2}} = \frac{d+2}{\beta_1^{D=4}} \tag{118}$$

where β_1^D is the beta function for the D-dimensional theory. For $d = 2$ i.e non supersymmetric case this equation have in general no solution.

5.2 Renormalons and Matrix Models

Let us consider a formal perturbative expansion

$$\sum_n g^{2n} C^n n! \tag{119}$$

where C is given in terms of the beta function. This serie diverges as soon as the $N+1$ term becomes larger than the Nth. This take place when

$$g^2 C n = 1 \tag{120}$$

We can cutoff the perturbative series at the order determined by (120). The error introduced by this cutoff can be estimated as the value of $g^{2n}C^n n!$ for

$$n = \frac{1}{Cg^2} \tag{121}$$

We will identify this estimate as the *non perturbative contribution*. Using the asymptotic Stirling's formula

$$log\Gamma(n) = (n+\frac{1}{2})lnn - n + \frac{1}{2}ln2\pi + \sum_l \frac{B_{2l}}{2l(2l-1)n^{2l-1}} \tag{122}$$

we get in first approximation

$$g^{2n}C^n n! = g^{2n}C^n n^n e^{-n} = e^{-\frac{1}{g^2 C}} \tag{123}$$

where we have used (121). For $C = \frac{N}{8\pi^2}$ we get the typical fractional instanton exponential. Notice that if we use the whole expansion including the Bernoulli numbers we will get

$$e^{-\frac{1}{g^2 C} + \sum_l \frac{B_{2l}(g^2C)^{2l-1}}{2l(2l-1)}} \tag{124}$$

that can be probably interpreted as a hidden gravitational correction to the fractional instanton contribution. Let us define F_{np} as

$$F_{np}(n) = ln(g^{2n}C^n n!) \sim ln((ng^2C)^n.e^{-n}) \tag{125}$$

estimated at $n > \frac{1}{Cg^2}$ using Stirling's formula. Now we look for a sort of *"prepotential"* for F_{np} i.e some functional $\Phi(n)$ such that

$$\frac{\partial \Phi(n)}{\partial n} = F_{np}(n) \tag{126}$$

In the asymptotic regime the logarithm of Barnes G function is a natural candidate for the *"prepotential"* $\Phi(n)$ [6]. In fact the asymptotic expansion of Barnes function is

$$lnG(1+n) = n^2(\frac{lnn}{2} - \frac{3}{4}) + \frac{ln(2\pi)}{2}n - \frac{1}{12}lnn + \zeta'(-1) - \sum \frac{B_{2k+2}}{4k(k+1)n^{2k}} \tag{127}$$

[6] This is very much related with Malmsten's formula.

Therefore

$$\frac{\partial \frac{1}{g^4C^2} lnG(1+ng^2C)}{\partial n} \sim n(ln(ng^2C)-1) \sim F_{np}(n) \tag{128}$$

which means that we can take as "prepotential" $\Phi(n)$ for the non perturbative contribution $\frac{1}{g^4C^2}lnG(1+ng^2C)$.

The logarithm of Barnes $G(1+\hat{N})$ function is intimately related with the partition function of $U(\hat{N})$ Chern-Simons [11] on S^3 and with the gaussian Matrix model for $\hat{N}\times\hat{N}$ matrices [22], [23]. In both cases the contribution of factorials comes from the volume of $U(\hat{N})$ in the large $\hat{N}$ limit. Moreover the *genus* expansion of the logarithm of Barnes G function coincides with the free energy of Penner model [24].

What we have observed in the previous exercise is that the natural connection with four dimensional supersymmetric gauge theories is essentially due to the fact that the most natural estimate of the non perturbative effects comes from the contribution of the $n!$ (i.e generic renormalon effects) in the perturbative expansion. Notice that this can be done independently if the theory is or not supersymmetric. However supersymmetry could be crucial in order to relate the $n!$ i.e the Γ function of the perturbative expansion with Barnes G function that play the formal rol of a prepotential. A much more deep analysis of this relation [25] is necessary but all this seems to indicate that some topological field theories or $c=1$ Matrix models are playing the role of bookkeepings of the $n!$ renormalon divergences of asymptotically free massless field theories.

5.2.1 Remark: Hidden Gravity?

The logic of the previous section was based on estimating the error when we cutoff the perturbative expansions even in the very weak coupling regime. In asymptotically free theories and when we consider typical renormalon divergences as we did in previous section we find quite naturally what looks as a gravitational contribution, namely

$$e^{-\frac{1}{g^2C}(1-\sum_l \frac{B_{2l}(g^2C)^{2l}}{2l(2l-1)})} \tag{129}$$

In the IR renormalon case with C determined by the beta function we get ,in the $N=1$ supersymmetric case, a formal expansion in t'Hooft coupling t. The relevant *"gravitational"* correction is given by

$$\sum_l \frac{B_{2l}(t)^{2l-1}}{2l(2l-1)} \tag{130}$$

As a formal infinite series this is highly divergent even for very small t due to the grow of the Bernoulli numbers as $2l!$. This is similar to the familiar situation in Matrix $c=1$ models ([26]).

[7]. Notice that in (127) we really have $\sum^m \frac{B_{2k+2}}{4k(k+1)n^{2k}} + O(\frac{1}{n^{2m+2}})$.

We can also consider divergences of *instanton* type if we are not working in the planar limit. In this case the constant C is just a number of the order $8\pi^2$ and (130) becomes a expansion in g^2_{YM} i.e in string coupling constant similar to the standard genus expansion.

In summary we observe that for asymptotically free theories the uncertainties due to renormalon (and instanton) divergences in the perturbative expansion can be interpreted as associated with a hidden gravitational sector in the sense of Matrix $c = 1$ models. Many years ago t' Hooft was suggesting that asymptotic freedom is probably not enough [27]. He was refering to the uncertainties in the perturbative expansion even for arbitrarily small coupling. It looks that the new ingredient we need to fix these uncertainties is intimately related with the gravitational sector hidden in asymptotically free gauge theories.

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